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Some consequences of the van Cittert-Zernike theorem for partially polarized, stochastic electromagnetic fields

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Coherence and polarization properties of the field generated by a partially polarized incoherent source are examined in terms of the 2×2 cross-spectral density matrix, with a view to deriving an electromagnetic counterpart of the van Cittert-Zernike theorem. Emphasis is placed on the polarization properties of the field. As some consequences of the extended theorem, we found that the spectral degree of polarization of the resultant field is generally different from that of the source, and that both are the same in a special case. A different, somewhat unconventional view of the extended theorem is also discussed in brief. © 2009 Optical Society of America

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Over the last few years there has been growing interest in the studies of coherence and polarization of stochastic electromagnetic fields (see, for example, [1–8]). In particular, polarization properties of partially coherent electromagnetic beams on propagation have been thoroughly investigated with the help of the propagation law for the 2×2 cross-spectral density matrix [9–14]. It is known, in general, that the two concepts of coherence and polarization properties of electromagnetic beams may be examined in a unified manner with this correlation matrix [3] (and also with the related correlation matrix under certain conditions [1]).

In the coherence theory of scalar fields, propagation laws for the correlation functions such as the cross-spectral density function have played a major role. They describe how the correlation properties of the field change on propagation. In a special case when the initial source is spatially incoherent, they lead to an important theorem, known as the van Cittert-Zernike theorem, showing that the spatial coherence is gained by the process of propagation ([15], Sec.3.2). It is to be noted that this theorem for scalar fields does not provide any information about the polarization properties of the field.

A few attempts have so far been made to generalize the conventional van Cittert-Zernike theorem for scalar fields to its electromagnetic counterpart and consequently a number of interesting results have been derived. The pioneering study has been conducted by Gori *et al.* who extended the theorem in terms of the 2×2 mutual intensity matrix (called the beam coherence-polarization matrix [1]) and examined coherence and polarization properties of the field generated by an incoherent unpolarized source covered with a specific polarizing filter [2]. Subsequently, Alonso *et al.* formulated the generalized theorem in terms of the 3×3 cross-spectral density matrix, which makes it possible to characterize the correlation properties of the three-dimensional field [6]. Very recently, Ostrovsky *et al.* dealt with the extended theorem formulated in terms of the 2×2 cross-spectral density matrix and showed the explicit expressions for the spectral degree of coherence and that of polarization of the field generated by an incoherent electromagnetic source [8]. However, the definition of the incoherent source in this study is somewhat different from that adopted in the two previous studies. Furthermore, the spectral degree of coherence of the electromagnetic field is defined in accordance with the definition proposed by Tervo *et al.* [4].

In this Letter, we describe an extension of the van Cittert-Zernike theorem in terms of the 2×2 cross-spectral density matrix to examine coherence and polarization properties of the field generated by a partially polarized, incoherent electromagnetic source. In our treatment, the spectral degree of coherence is introduced in accordance with the definition proposed by Wolf [3] for the reason stated in the footnote 7 of Ref. [7]. As some important consequences which have not been presented in the previous studies, special emphasis is placed on the relationships between the spectral degree of polarization of the source and that of the field which it generates.

Let us start with the paraxial propagation of the 2×2 cross-spectral density matrix in free space from a plane $z = 0$ to the half space $z > 0$. It may be expressed by the formula ([15], Sec.9.4.1)

$$\mathbf{W}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \iint_{z=0} \mathbf{W}^{(0)}(\boldsymbol{\rho}_1', \boldsymbol{\rho}_2', \omega) G^*(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1', \omega) G(\boldsymbol{\rho}_2, \boldsymbol{\rho}_2', \omega)\, \mathrm{d}^2\rho_1' \mathrm{d}^2\rho_2', \tag{1}$$

where $\mathbf{W}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ and $\mathbf{W}^{(0)}(\boldsymbol{\rho}_1', \boldsymbol{\rho}_2', \omega)$ are the 2×2 cross-spectral density matrices in the planes $z > 0$ and $z = 0$, respectively, and the Green's function $G(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)$ is given by the expression

$$G(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega) = -\frac{\mathrm{i}k}{2\pi z} \exp\left[\frac{\mathrm{i}k}{2z}(\boldsymbol{\rho} - \boldsymbol{\rho}')^2\right] \tag{2}$$

with $k = \omega/c$, ω being the frequency and c being the speed of light in vacuum. The asterisk denotes the complex conjugate. The elements of $\mathbf{W}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ are given by the expression

$$W_{ij}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \langle E_i^*(\boldsymbol{\rho}_1, \omega) E_j(\boldsymbol{\rho}_2, \omega)\rangle, \qquad (i = x, y;\ j = x, y), \tag{3}$$

where $E_x(\boldsymbol{\rho},\omega)$ and $E_y(\boldsymbol{\rho},\omega)$ are the components of the random electric field in the x and y directions, respectively, at a point specified by a two-dimensional position vector $\boldsymbol{\rho}$ and the angular brackets denote the ensemble average. The elements of $\mathbf{W}^{(0)}(\boldsymbol{\rho}'_1,\boldsymbol{\rho}'_2,\omega)$ may be written in a strictly similar form.

Suppose now that a partially polarized, incoherent electromagnetic source occupying a domain D is located in the plane $z=0$. Such a source may be characterized by the 2×2 cross-spectral density matrix of the form [2,6]

$$\mathbf{W}^{(0)}(\boldsymbol{\rho}'_1,\boldsymbol{\rho}'_2,\omega)=\mathbf{S}^{(0)}(\boldsymbol{\rho}'_1,\omega)\delta(\boldsymbol{\rho}'_2-\boldsymbol{\rho}'_1), \tag{4}$$

where $\delta(\boldsymbol{\rho}'_2-\boldsymbol{\rho}'_1)$ denotes the two-dimensional Dirac delta function and

$$\mathbf{S}^{(0)}(\boldsymbol{\rho}',\omega)=[S^{(0)}_{ij}(\boldsymbol{\rho}',\omega)], \qquad (i=x,y;\ j=x,y), \tag{5}$$

is the spectral counterpart of the ordinary polarization matrix (for short, we call it the spectral polarization matrix below) of the source. On substituting from Eqs.(2) and (4) into Eq.(1) and performing the integration with respect to $\boldsymbol{\rho}'_2$, we obtain the expression

$$\mathbf{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\left(\frac{k}{2\pi z}\right)^2\exp\left[\frac{\mathrm{i}k}{2z}(\boldsymbol{\rho}_2^2-\boldsymbol{\rho}_1^2)\right]\int_D \mathbf{S}^{(0)}(\boldsymbol{\rho}',\omega)\exp\left[-\frac{\mathrm{i}k}{z}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)\cdot\boldsymbol{\rho}'\right]\mathrm{d}^2\rho'. \tag{6}$$

This is the non-normalized version of the electromagnetic counterpart of the van Cittert-Zernike theorem under the paraxial approximation. Using this formula, we obtain for the spectral degree of coherence of the resultant field the expression ([15], Sec.9.2)

$$\begin{aligned}\eta(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)&\equiv\frac{\mathrm{tr}\,\mathbf{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)}{\sqrt{\mathrm{tr}\,\mathbf{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_1,\omega)}\sqrt{\mathrm{tr}\,\mathbf{W}(\boldsymbol{\rho}_2,\boldsymbol{\rho}_2,\omega)}}\\&=\frac{\exp\left[\frac{\mathrm{i}k}{2z}(\boldsymbol{\rho}_2^2-\boldsymbol{\rho}_1^2)\right]\int_D S^{(0)}(\boldsymbol{\rho}',\omega)\exp\left[-\frac{\mathrm{i}k}{z}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)\cdot\boldsymbol{\rho}'\right]\mathrm{d}^2\rho'}{\int_D S^{(0)}(\boldsymbol{\rho}',\omega)\mathrm{d}^2\rho'},\end{aligned} \tag{7}$$

where $S^{(0)}(\boldsymbol{\rho}',\omega)=\mathrm{tr}\,\mathbf{S}^{(0)}(\boldsymbol{\rho}',\omega)$ is the spectral density (intensity at frequency ω) across the source and tr denotes the trace. Equation (7) may be interpreted as a natural generalization of the conventional van Cittert-Zernike theorem, since it clearly shows that the modulus of the spectral degree of coherence of the resultant electromagnetic field is equal to the modulus of the normalized Fourier transform of the intensity distribution across the electromagnetic source.

The present definition of the spectral degree of coherence of the electromagnetic beam is based on the visibility of interference fringes produced in Young's experiment [3]. On the other hand, the spectral degree of coherence defined in Ref. [4] and employed in the closely related study [8] is given as the normalized Frobenius norm of the cross-spectral density matrix, so that it is invariant under transformations into orthogonal curvilinear coordinate systems. However, this quantity may be more suitably referred to as the spectral degree of *correlation* since it describes the correlations between the components of the field (i.e., including both concepts of coherence and polarization), but it is not directly connected with the visibility of the interference fringe.

Such a difference in the definition of the spectral degree of coherence necessarily yields the difference in the resultant formula for the generalized van Cittert-Zernike theorem (cf. Eq.(24) of Ref. [8]). In this connection, it is worth noting that some other definitions on the electromagnetic degree of coherence have also been proposed recently, though they represent actually the spectral degree of correlation in the present sense [5, 16].

We next consider the spectral degree of polarization of the resultant field, given by the expression ([15], Sec.9.2)

$$P(\boldsymbol{\rho},\omega) = \sqrt{1 - \frac{4\det \mathbf{W}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)}{[\mathrm{tr}\,\mathbf{W}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)]^2}}, \tag{8}$$

where det denotes the determinant and $\mathbf{W}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)$ is the spectral polarization matrix obtained by setting $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2 = \boldsymbol{\rho}$ (say) in the 2×2 cross-spectral density matrix $\mathbf{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)$. Using Eqs.(8) and (5) (see also Eq.(4)), we readily find that the spectral degree of polarization of the source is given by the expression

$$P^{(0)}(\boldsymbol{\rho}',\omega) = \frac{\sqrt{[S^{(0)}_{xx}(\boldsymbol{\rho}',\omega) - S^{(0)}_{yy}(\boldsymbol{\rho}',\omega)]^2 + 4|S^{(0)}_{xy}(\boldsymbol{\rho}',\omega)|^2}}{S^{(0)}_{xx}(\boldsymbol{\rho}',\omega) + S^{(0)}_{yy}(\boldsymbol{\rho}',\omega)}. \tag{9}$$

On substituting from Eq.(6) into Eq.(8) and using Eq.(5), we obtain for the spectral degree of polarization of the resultant field the expression

$$P(\boldsymbol{\rho},\omega) = \frac{\sqrt{\left\{\int_D [S^{(0)}_{xx}(\boldsymbol{\rho}',\omega) - S^{(0)}_{yy}(\boldsymbol{\rho}',\omega)]\mathrm{d}^2\rho'\right\}^2 + 4\left|\int_D S^{(0)}_{xy}(\boldsymbol{\rho}',\omega)\mathrm{d}^2\rho'\right|^2}}{\int_D [S^{(0)}_{xx}(\boldsymbol{\rho}',\omega) + S^{(0)}_{yy}(\boldsymbol{\rho}',\omega)]\mathrm{d}^2\rho'}. \tag{10}$$

It follows from Eqs.(9) and (10) that the spectral degree of polarization of the resultant field is, in general, different from that of the source. Furthermore, the spectral degree of polarization of the resultant field is necessarily independent of the location $\boldsymbol{\rho}$ in the resultant field, even though the spectral degree of polarization of the source depends generally on the source point $\boldsymbol{\rho}'$. In fact, an interesting example of the changes in the degree of polarization on propagation was given by Gori *et al.* [2], who showed that the degree of polarization of the field produced by a specific source with a position-dependent polarization matrix, but with the degree of polarization being unity at every source point is zero in the far field of the source, namely, the field produced by the completely polarized source becomes completely unpolarized on propagation.

At the first glance, these results seem to be inconsistent with those obtained by Ostrovsky *et al.* [8], who concluded that the spectral degree of polarization of the field generated by the incoherent electromagnetic source does not change on propagation in free space. This is not the case, however. Equation (10) clearly shows the invariance of the spectral degree of polarization of the resultant field on propagation in the sense that the spectral degree of polarization of the resultant field in the plane $z_1 > 0$ is equivalent to that in the different plane $z_2 > 0$ $(z_1 \neq z_2)$.

As a special case, we assume that the spectral polarization matrix $\mathbf{S}^{(0)}(\boldsymbol{\rho}', \omega)$ of the source is uniform across the source plane, i.e., it is independent of the location $\boldsymbol{\rho}'$ in the source plane. Using new symbols $S_{ij}^{(0)}(\omega) \equiv S_{ij}^{(0)}(\boldsymbol{\rho}', \omega)$, $(i = x, y; j = x, y)$ for the elements of $\mathbf{S}^{(0)}(\boldsymbol{\rho}', \omega)$, Eqs.(9) then becomes

$$P^{(0)}(\boldsymbol{\rho}', \omega) = \frac{\sqrt{[S_{xx}^{(0)}(\omega) - S_{yy}^{(0)}(\omega)]^2 + 4|S_{xy}^{(0)}(\omega)|^2}}{S_{xx}^{(0)}(\omega) + S_{yy}^{(0)}(\omega)}. \tag{11}$$

In a similar way, it is not difficult to show that Eq.(10) reduces to the same functional form as Eq.(11). Accordingly, we find that the spectral degree of polarization of the resultant field is the same as that of the source, provided that the spectral polarization matrix of the source is uniform across the source plane.

The (spectral) degree of polarization of the electromagnetic beam is defined as the ratio of the intensity of the polarized portion to the total intensity and unambiguously given by Eq.(8). Therefore, in contrast to the electromagnetic degree of coherence, the definition of the (spectral) degree of polarization is expected to be identical everywhere.

Finally, it is of interest to provide a different, somewhat unconventional view of the extended van Cittert-Zernike theorem that we have derived in this Letter. As we have just seen, the 2×2 cross-spectral density matrices are employed to characterize stochastic electromagnetic sources and beams. It is known, in general, that these correlation matrices cannot be chosen at will and must be non-negative definite so that they are physically realizable [17]. A sufficient condition for these matrices to satisfy the non-negative definiteness condition has recently been derived [18]. Interestingly, the functional form of the sufficient condition is essentially the same as Eq.(1) with Eq.(4), except that the (Green's) function $G(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)$ given by Eq.(2) in the present study may be arbitrary and that the function $G(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)$ for the x component of the electric field ($G_x(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)$, say), i.e.,

$$E_x(\boldsymbol{\rho}, \omega) = \int_{z=0} E_x^{(0)}(\boldsymbol{\rho}', \omega) G_x(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)\, \mathrm{d}^2\rho', \tag{12}$$

and the corresponding function for the y component ($G_y(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)$, say) may be, in general, different from each other. In this sense, the electromagnetic counterpart of the van Cittert-Zernike theorem given by Eq.(6) may be regarded as a special case of that condition.

In summary, we have derived the extended van Cittert-Zernike theorem for partially polarized, stochastic electromagnetic fields. As a result, we have found that the spectral degree of coherence of the field generated by a partially polarized, incoherent electromagnetic source is obtained in a manner strictly similar to the conventional van Cittert-Zernike theorem. We have also found that the spectral degree of polarization of the resultant field differs, in general, from that of the source, but both are the same in a special case when the spectral polarization matrix of the source is independent of the position across the source plane. The different view of the extended theorem discussed briefly in the final part of this Letter may provide some insight into the underlying physics of the extended theorem.

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